A Wick-rotatable metric is purely electric

Christer Helleland and **Sigbjørn Hervik**

Faculty of Science and Technology,
University of Stavanger,
N-4036 Stavanger, Norway

christer.helleland@uis.no
sigbjorn.hervik@uis.no

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Abstract

We show that a metric of arbitrary dimension and signature which allows for a standard Wick-rotation to a Riemannian metric necessarily has a purely electric Riemann and Weyl tensor.

1 Introduction

In quantum theories a Wick-rotation is a mathematical trick to relate Minkowski space to Euclidean space by a complex analytic extension to imaginary time. This enables us to relate a quantum mechanical problem to a statistical mechanical one relating time to the inverse temperature. This trick is highly successful and is used in a wide area of physics, from statistical and quantum mechanics to Euclidean gravity and exact solutions.

In spite of its success, there is a question about its range of applicability. A question we can ask is: *Given a spacetime, does there exist a Wick-rotation to transform the metric to a Euclidean one?*

Here we will give a partial answer to this question and will give a necessary condition for a Wick-rotation (as defined below) to exist. However, before we prove our main theorem, we need to be a bit more precise with what we mean by a Wick-rotation. Consider a pseudo-Riemannian metric (of arbitrary dimension and signature). We need to allow for more general coordinate transformations than the real diffeomorphisms preserving the metric signature – namely to complex analytic continuations of the real metric [1, 2] .

Consider a point p and a neighbourhood, U, of p. Assume this nighbourhood is an analytic neighbourhood and that x^μ are coordinates on U so that $x^\mu \in \mathbb{R}^n$. We will adapt the coordinates to the point p so that p is at the origin of this coordinate system. Consider now the complexification of $x^\mu \mapsto x^\mu + iy^\mu = z^\mu \in \mathbb{C}^n$. This complexification enables us to consider the complex analytic neighbourhood $U^{\mathbb{C}}$ of p.

Furthermore, let $g^{\mathbb{C}}_{\mu\nu}$ be a complex bilinear form (a holomorphic metric)

induced by the analytic extension of the metric:

$$g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu\mathrm{d}x^\nu \mapsto g^{\mathbb{C}}_{\mu\nu}(z^\rho)\mathrm{d}z^\mu\mathrm{d}z^\nu.$$

Next, consider a real analytic submanifold containing p: $\bar{U} \subset U^{\mathbb{C}}$ with coordinates $\bar{x}^\mu \in \mathbb{R}^n$. The imbedding $\iota : \bar{U} \mapsto U^{\mathbb{C}}$ enables us to pull back the complexified metric $\boldsymbol{g}^{\mathbb{C}}$ onto $\bar{U}$:

$$\bar{\boldsymbol{g}} \equiv \iota^*\boldsymbol{g}^{\mathbb{C}}. \tag{1}$$

In terms of the coordinates $\bar{x}^\mu$: $\bar{\boldsymbol{g}} = \bar{g}_{\mu\nu}(\bar{x}^\rho)\mathrm{d}\bar{x}^\mu\mathrm{d}\bar{x}^\nu$. This bilinear form may or may not be real. However, *if the bilinear form $\bar{g}_{\mu\nu}(\bar{x}^\rho)\mathrm{d}\bar{x}^\mu\mathrm{d}\bar{x}^\nu$ is real (and non-degenerate) then we will call it an analytic extension of $g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu\mathrm{d}x^\nu$ with respect to p,* or simply *a Wick-rotation* of the real metric $g_{\mu\nu}(x^\rho)\mathrm{d}x^\mu\mathrm{d}x^\nu$. This clearly generalises the concept of Wick-rotations from the standard Minkowskian setting to a more general setting [3].

In the following, let us call the Wick-rotation, in the sense above, for $\bar{\phi}$; i.e., $\bar{\phi} : U \to \bar{U}$. We note that this transformation is complex, and we can assume, since U is real analytic, that $\bar{\phi}$ is analytic.

The Wick-rotation in the sense above, leaves the point p stationary. It therefore induces a linear transformation, M, between the tangent spaces T_pU and $T_p\bar{U}$. The transformation M is complex and therefore may change the metric signature; consequently, even if the metric $\bar{g}_{\mu\nu}$ is real, it does not necessarily need to have the same signature of $g_{\mu\nu}$.

Consider now the curvature tensors, R and $\nabla^{(k)}R$ for $g_{\mu\nu}$, and $\bar{R}$ and $\bar{\nabla}^{(k)}\bar{R}$ for $\bar{g}_{\mu\nu}$. Since both metrics are real, their curvature tensors also have to be real. The analytic continuation, in the sense above, induces a linear transformation of the tangent spaces; consequently, this would relate the Riemann tensors R and $\bar{R}$ through a complex linear transformation. It is useful to introduce an orthonormal frame $\mathbf{e}_\mu$. The orthonormal frames $\mathbf{e}_\mu$ and $\bar{\mathbf{e}}_\mu$ are related through their complexified frame $\mathbf{e}^{\mathbb{C}}_\mu$. We can define a complex orthonormal frame requiring the inner product[1] $\left\langle \mathbf{e}^{\mathbb{C}}_\mu, \mathbf{e}^{\mathbb{C}}_\nu \right\rangle = \delta_{\mu\nu}$. This inner product is invariant under the complex orthogonal transformations, $O(n,\mathbb{C})$. The real frames $\mathbf{e}_\mu$ and $\bar{\mathbf{e}}_\mu$ are obtained by restricting the complex frame. As an example, consider the standard holomorphic inner product space $(\mathbb{C}^n, \boldsymbol{g}^{\mathbb{C}}_0)$ and $(\mathbf{e}^{\mathbb{C}}_1, ..., \mathbf{e}^{\mathbb{C}}_n)$ the standard basis. Then a real subspace is $V = \mathrm{span}_{\mathbb{R}}(i\mathbf{e}^{\mathbb{C}}_1, ..., i\mathbf{e}^{\mathbb{C}}_p, \mathbf{e}^{\mathbb{C}}_{p+1}, ..., \mathbf{e}^{\mathbb{C}}_n)$, and the corresponding metric (obtained from $\boldsymbol{g}^{\mathbb{C}}_0$ by restriction) is real. All such real subspaces V (of different signatures) are obtained from such identifications and hence different real subspaces V are related via the action of the complex orthogonal group $O(n,\mathbb{C})$ (for more details, see e.g. [4, 5]).

Hence, we consider the real vector spaces T_pU and $T_p\bar{U}$ as embedded in the complexified vector space $(T_pU)^{\mathbb{C}} \cong (T_p\bar{U})^{\mathbb{C}}$. The real frames are thus related though a restriction of a complex frame having an $O(n,\mathbb{C})$ structure group. If moreover the tangent spaces T_pU and $T_p\bar{U}$ are embedded:

$$T_pU, T_p\bar{U} \hookrightarrow (T_pU)^{\mathbb{C}} \cong (T_p\bar{U})^{\mathbb{C}},$$

such that they form a *compatible triple* [2], then we shall say that the real submanifolds: U and $\bar{U}$, are related through a *standard Wick-rotation.* A standard

[1]This is a not really a proper inner product since it is not positive definite, but rather a $\mathbb{C}$-bilinear non-degenerate form defining a *holomorphic inner product.*

[2]Let W and $\widetilde{W}$ be real slices of a holomorphic inner product space: (E, g). Assume they

Wick-rotation allows us to choose commuting Cartan involutions of the real metrics.

Note the special case where $\bar{U}$ is Riemannian, then the condition of being Wick-rotated by a standard Wick-rotation, is just the condition that the conjugation maps of T_pU, and $T_p\bar{U}$ must commute when embedded into $(T_pU)^{\mathbb{C}}$, i.e T_pU and $T_p\bar{U}$ are compatible real forms.

We refer to the manuscript [5], for more details about standard Wick-rotations and the connection with real GIT, and the special case of $\bar{U}$ being Riemannian.

By using $\bar{\phi}$ we can relate the metrics $\boldsymbol{g} = \bar{\phi}^*\bar{\boldsymbol{g}}$. Since the map is analytic (albeit complex), the curvature tensors are also related via $\bar{\phi}$. If R and $\bar{R}$ are the Riemann curvature tensors for U and $\bar{U}$ respectively, then these are related, using an orthonormal frame, via an $O(n,\mathbb{C})$ transformation. Consider the components of the Riemann tensor as a vector in some $\mathbb{R}^N \subset \mathbb{C}^N$. If there exists a Wick-rotation of the metric at p, then the (real) Riemann curvature tensors of U and $\bar{U}$ must be real restrictions of vectors that lie in the same $O(n,\mathbb{C})$ orbit in $\mathbb{C}^N$.

Note: This definition of a Wick-rotation does not include the more general analytic continuations defined by Lozanovski [6]. In particular, we consider *one* particular metric (thus not a family of them) and we require that the point p is fixed and is therefore more of a complex rotation.

In the following we will utilise the study of real orbits of semi-simple groups, see e.g. [7, 8]. In particular, the considerations made in [9] will be useful. For a more general introduction to the structure of Lie algebras including the Cartan involution, see, for example [10, 11].

2 The electric/magnetic parts of a tensor

Following [9], we can introduce the electric and magnetic parts of a tensor by considering the eigenvalue decomposition of the tensor under the Cartan involution θ of the real Lie algebras $\mathfrak{o}(p,q)$. This involution can be extended to all tensors, and to vectors $\mathbf{v} \in T_pM$ in particular. Considering an orthonormal frame, so that:

$$\boldsymbol{g}(\mathbf{e}_\mu, \mathbf{e}_\mu) = \begin{cases} -1, & 1 \leq \mu \leq p \\ +1 & p+1 \leq \mu \leq p+q = n, \end{cases}$$

the $\theta : T_pM \to T_pM$, can be defined as the linear operator:

$$\theta(\mathbf{e}_\mu) = \begin{cases} -\mathbf{e}_\mu, & 1 \leq \mu \leq p \\ +\mathbf{e}_\mu & p+1 \leq \mu \leq p+q = n. \end{cases}$$

are both real forms of $W^{\mathbb{C}} \subset (E, g)$. Let V be another real slice of E, and a real form of $W^{\mathbb{C}}$, with Euclidean signature. Suppose $W, \widetilde{W}$ and V are pairwise compatible (i.e their conjugation maps commute pairwise), then a triple: $(W, \widetilde{W}, V)$, will be called a *compatible triple*. **Examples**: $\left(\mathbb{R} \oplus i\mathbb{R}, i\mathbb{R} \oplus \mathbb{R}, \mathbb{R}^2\right)$ with $E := \mathbb{C}^2$, and $\left(\mathfrak{o}(p,q), \mathfrak{o}(\tilde{p},\tilde{q}), \mathfrak{o}(n)\right)$ with $E := \mathfrak{o}(n,\mathbb{C})$ and $g := \kappa(-,-)$ (the Killing form).

Clearly, this implies that the bilinear map:

$$\langle X,Y\rangle_\theta := \boldsymbol{g}(\theta(X),Y), \qquad X,Y\in T_pM$$

defines a positive definite inner-product on T_pM. This Cartan involution can be extended tensorially to arbitrary tensor products.

Given a Cartan involution θ, then since $\theta^2=\mathrm{Id}$, its eigenvalues are ± 1 and any tensor T has an eigenvalue decomposition:

$$T = T_+ + T_-, \qquad \text{where} \;\; \theta(T_\pm)=\pm T_\pm.$$

A space is called *purely electric* (PE) if there exists a Cartan involution so that the Weyl tensor decomposes as $C=C_+$ [9]. Furthermore, a space is called *purely magnetic* (PM) if the Weyl tensor decomposes as $C=C_-$. If this property occurs also for the Riemann tensor, we call the space Riemann purely electric (RPE) or magnetic (RPM), respectively. Clearly, RPE implies PE.

3 The Riemann curvature operator

The Riemann curvature tensor can (pointwise) be seen as a bivector operator:

$$\mathrm{Riem}: \wedge^2\Omega_p(M)\to\wedge^2\Omega_p(M).$$

In a pseudo-Riemannian space of signature (p,q) the metric $\boldsymbol{g}$ will provide an isomorphism between the space of bivectors, $\wedge^2\Omega_p(M)$, and the Lie algebra $\mathfrak{g}=\mathfrak{o}(p,q)$. This can be seen as follows. The Lie algebra $\mathfrak{o}(p,q)$ is defined through the action of $O(p,q)$ on the tangent space T_pM: For any $G\in O(p,q)$, $G: T_pM\to T_pM$ so that $\boldsymbol{g}(G\cdot v, G\cdot u)=\boldsymbol{g}(v,u)$ for all $v,u\in T_pM$. Using the exponential map $\exp:\mathfrak{o}(p,q)\to O(p,q)$, we get the requirement that $\boldsymbol{g}(X(v),u)+\boldsymbol{g}(v,X(u))=0$ for any $X\in\mathfrak{o}(p,q)$. Consequently, X is antisymmetric with respect to the metric $\boldsymbol{g}$. In terms of the basis vectors, we can write $X=(X^\mu{}_\nu)$ and the antisymmetry condition implies that by raising an index we get $X^{\mu\nu}=-X^{\nu\mu}$ and can therefore be considered as a bivector. Since the dimensions match, the metric thus provides with an isomorphism between the Lie algebra $\mathfrak{o}(p,q)$ and the space of bivectors $\wedge^2\Omega_p(M)$ at a point[3].

Consequently, the Riemann curvature operator can also be viewed as an endomorphism of $V:=\mathfrak{g}$ treated as a vector space. Consider therefore any $\mathsf{R}\in\mathrm{End}(V)$:

$$\mathsf{R}: V\to V.$$

This endomorphism can be split in a symmetric and anti-symmetric part, $\mathsf{R}=\mathsf{S}+\mathsf{A}$, with respect to the metric induced by $\boldsymbol{g}$ (which we also will call $\boldsymbol{g}$ and is proportional to the Killing form κ on V)[4]:

$$\boldsymbol{g}(\mathsf{S}(x),y)=\boldsymbol{g}(\mathsf{S}(y),x), \quad \boldsymbol{g}(\mathsf{A}(x),y)=-\boldsymbol{g}(\mathsf{A}(y),x) \qquad \forall x,y\in\mathfrak{g}.$$

[3]Indeed, this is a mere consequence of the fact that there is an $O(p,q)$-module isomorphism between T_pM and T^*_pM.

[4]That the metric induced by $\boldsymbol{g}$ is proportional to the Killing form can be seen either by explicit computation, or from considering κ as a even-ranked tensor over $V^*\otimes V^*$ which is invariant under the action of $O(p,q)$. By, e.g., section 5.3.2 in [12], this tensor is necessarily proportional to the metric tensor on V induced by $\boldsymbol{g}$.

This metric is invariant under the Lie group action of $G = O(p,q)$:

$$\boldsymbol{g}(h\cdot x, h\cdot y) = \boldsymbol{g}(x,y),$$

where $h\cdot x$ is the natural Lie group action on the Lie algebra given by the adjoint: $h\cdot x := \mathrm{Ad}_h(x) = h^{-1}xh$.

Consider now a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Then we define the inner-product on $V = \mathfrak{g}$ as follows:

$$\langle x, y\rangle_\theta = \boldsymbol{g}(\theta(x), y),$$

which is just proportional to $\kappa_\theta(-,-) := -\kappa(-,\theta(-))$. We can now, similarly, split any $\mathsf{R} \in \mathrm{End}(V)$ in a symmetric and anti-symmetric part, $\mathsf{R} = \mathsf{R}_+ + \mathsf{R}_-$, with respect to the inner-product $\langle -,-\rangle_\theta$:

$$\langle \mathsf{R}_+(x), y\rangle_\theta = \langle \mathsf{R}_+(y), x\rangle_\theta\,, \quad \langle \mathsf{R}_-(x), y\rangle_\theta = -\langle \mathsf{R}_-(y), x\rangle_\theta\,, \qquad \forall x,y\in\mathfrak{g}.$$

We shall denote $V = \mathfrak{k}\oplus\mathfrak{p}$, for the Cartan decomposition w.r.t θ, where $\mathfrak{k}$ is the compact part and $\mathfrak{p}$ is the non-compact part.

Suppose now that the real submanifolds U and $\bar{U}$ are two Wick-rotatable spaces (of the same dimension) by a standard Wick-rotation at a fixed intersection point p, but with one of the real slices being Riemannian. So we can set $V := \mathfrak{o}(p,q)$ as before, and introduce (similarly as with V above), $\tilde{V} := \mathfrak{o}(n)$, a compact real form of $V^{\mathbb{C}} := \mathfrak{o}(n,\mathbb{C})$. These real forms V and $\tilde{V}$, will naturally be compatible when embedded into $V^{\mathbb{C}}$, w.r.t to a standard Wick-rotation, i.e it lets us fix a Cartan involution θ, such that $\mathfrak{k} = V\cap\tilde{V}$, and $\mathfrak{p} = V\cap i\tilde{V}$. Again we refer to the paper [5] for details.

The space of endomorphisms, $\mathrm{End}(V)$, is also a vector space with the group action given by conjugation:

$$(g\cdot X)(v) := gX(g^{-1}vg)g^{-1}, \quad X\in\mathrm{End(V)},\ v\in V,\ g\in G.$$

Call this action ρ. We can thus define $\mathcal{V} := \mathrm{End}(V)$, and extend the Cartan involution, θ, as well as $\boldsymbol{g}$ tensorially to $\mathcal{V}$. We define analogously an inner product on $\mathcal{V}$:

$$\langle\!\langle X, Y\rangle\!\rangle_\theta = \boldsymbol{g}(\theta(X), Y), \qquad X, Y\in\mathcal{V}.$$

The inner product can assume to have the following properties (see [7]) w.r.t the action ρ:

1. The inner product is K-invariant, where $K \cong O(p)\times O(q)$ is the maximally compact subgroup of G with Lie algebra $\mathfrak{k}$.
2. $d\rho(\mathfrak{k}) : \mathcal{V}\to\mathcal{V}$ consists of skew-symmetric maps w.r.t $\langle\!\langle X, Y\rangle\!\rangle_\theta$.
3. $d\rho(\mathfrak{p}) : \mathcal{V}\to\mathcal{V}$ consists of symmetric maps w.r.t $\langle\!\langle X, Y\rangle\!\rangle_\theta$.

With such an inner product, enables us to apply the results in [7], i.e we can make use of minimal vectors for determining the closure of real orbits.

Defining $\tilde{\mathcal{V}} := \mathrm{End}(\tilde{\mathrm{V}})$ similarly, we have $\mathcal{V}, \tilde{\mathcal{V}} \subset \mathcal{V}^{\mathbb{C}}$ where $\mathcal{V}^{\mathbb{C}} := \mathrm{End}(V^{\mathbb{C}})$. Now since V and $\tilde{V}$ are real forms of $V^{\mathbb{C}}$ then $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are real forms of $\mathcal{V}^{\mathbb{C}}$.

This is seen in the following way. A map $\mathsf{R} \in \mathcal{V}$ can be extended to the complex linear map $\mathsf{R}^{\mathbb{C}} \in \mathcal{V}^{\mathbb{C}}$ by defining:

$$\mathsf{R}^{\mathbb{C}}(x+iy) := \mathsf{R}(x) + i\mathsf{R}(y), \quad x, y \in V.$$

So we view a map R as the complex linear map $\mathsf{R}^{\mathbb{C}}$. Thus regard $\tilde{\mathcal{V}}$ like this as well. We shall just write R instead of $\mathsf{R}^{\mathbb{C}}$.

We thus assume we have two endomorphisms (the Riemann curvature operators): $\mathsf{R} : V \to V$ (arbitrary pseudo-Riemannian), and $\tilde{\mathsf{R}} : \tilde{V} \to \tilde{V}$ (Riemannian). Now since we have the two real slices: U and $\bar{U}$, which are Wick-rotated at the point p, then necessarily $\mathsf{R} \in \mathcal{V}$ and $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$ must be conjugated by an element $g \in G^{\mathbb{C}} := O(n, \mathbb{C})$.

Set now $G := O(p,q)$ (with Lie algebra $V := \mathfrak{o}(p,q)$) and $\tilde{G} := O(n)$ (with Lie algebra $\tilde{V} := \mathfrak{o}(n)$) for the real forms embedded into $G^{\mathbb{C}}$ (with Lie algebra $V^{\mathbb{C}} := \mathfrak{o}(n, \mathbb{C})$) w.r.t a standard Wick-rotation[5]. Now we have a commutative diagram of conjugation actions:

$$\begin{array}{ccc} G^{\mathbb{C}} & \xrightarrow{\rho^{\mathbb{C}}} & GL(\mathcal{V}^{\mathbb{C}}) \\ {\scriptstyle i}\uparrow & & {\scriptstyle i}\uparrow \\ G & \xrightarrow{\rho} & GL(\mathcal{V}) \end{array} \tag{2}$$

Where $\rho^{\mathbb{C}}$ is also the action given by conjugation, where $G^{\mathbb{C}}$ is viewed as a real Lie group, and $\mathcal{V}^{\mathbb{C}}$ is also viewed as a real vector space. We similarly have such a diagram for the the group $\tilde{G}$, where the conjugation action: $\tilde{\rho}$, on $\tilde{\mathcal{V}}$ also extends to $\rho^{\mathbb{C}}$.

Now our real Riemann curvature operators from U and $\bar{U}$: R and $\tilde{\mathsf{R}}$, will now lie in the same complex orbit, i.e $G^{\mathbb{C}} \cdot \mathsf{R} = G^{\mathbb{C}} \cdot \tilde{\mathsf{R}}$.

So therefore in what follows, we will consider the real orbits, $G \cdot \mathsf{R}$, $\tilde{G} \cdot \tilde{\mathsf{R}}$ and its complexified orbit $G^{\mathbb{C}} \cdot \mathsf{R}$ defined by the conjugation action of the group on an endomorphism: $\mathsf{R} \in \mathcal{V}$ and $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$, as follows [7, 8, 9]

$$\begin{aligned} G \cdot \mathsf{R} &:= \{h \cdot \mathsf{R} \mid h \in O(p,q)\} \subset \mathcal{V} \\ \tilde{G} \cdot \tilde{\mathsf{R}} &:= \{h \cdot \tilde{\mathsf{R}} \mid h \in O(n)\} \subset \tilde{\mathcal{V}} \\ G^{\mathbb{C}} \cdot \mathsf{R} &:= \{h \cdot \mathsf{R} \mid h \in O(n, \mathbb{C})\} \subset \mathcal{V}^{\mathbb{C}}. \end{aligned}$$

Theorem 3.1. *Suppose* $\mathsf{R} = \mathsf{S} + \mathsf{A} \in \mathcal{V}$ *where* S, A *are the symmetric/antisymmetric parts w.r.t* $\boldsymbol{g}$ *respectively. Assume that there exists a (real)* $\tilde{\mathsf{R}} \in G^{\mathbb{C}} \cdot \mathsf{R}$ *so that* $\tilde{\mathsf{R}} \in \tilde{\mathcal{V}}$ *i.e we assume:* $G^{\mathbb{C}} \cdot \mathsf{R} = G^{\mathbb{C}} \cdot \tilde{\mathsf{R}}$. *Then there exists a Cartan involution* θ' *of* V *such that* $\mathsf{R}_+ = \mathsf{S}$ *and* $\mathsf{R}_- = \mathsf{A}$, *where* $\mathsf{R}_+, \mathsf{R}_-$ *are the symmetric/antisymmetric parts w.r.t* $\langle -, - \rangle_{\theta'}$ *respectively.*

Proof. Consider the orbits $G \cdot \mathsf{R}$ and $\tilde{G} \cdot \tilde{\mathsf{R}}$. Since the group $\tilde{G}$ is compact, the orbit $\tilde{G} \cdot \tilde{\mathsf{R}}$ is necessarily closed in $\tilde{\mathcal{V}}$; consequently, $G \cdot \mathsf{R}$, is closed as well

[5] G and $\tilde{G}$ are the structure groups of the real metrics restricting from the holomorphic metric, and thus consist of isometries: $T_pU \to T_pU$ and $T_p\bar{U} \to T_p\bar{U}$ of the real metrics respectively. These groups are naturally embedded into $O(n, \mathbb{C})$ as real forms, by complexification: $f \mapsto f^{\mathbb{C}}$.

and possesses a minimal vector[6] [7]. Denote by $\mathcal{M}(G^{\mathbb{C}}, \mathcal{V}^{\mathbb{C}})$ the set of minimal vectors in $\mathcal{V}^{\mathbb{C}}$. Assume that $X \in G \cdot \mathsf{R} \subset \mathcal{V}$ is minimal, then X is also a minimal vector in the complex orbit: $G^{\mathbb{C}} \cdot \mathsf{R}$. However since G and $\tilde{G}$ are compatible real forms (i.e V and $\tilde{V}$ are compatible[7]), and $\tilde{G}$ is a compact real form of $G^{\mathbb{C}}$, then necessarily:

$$G^{\mathbb{C}} \cdot \mathsf{R} \cap \mathcal{M}(G^{\mathbb{C}}, \mathcal{V}^{\mathbb{C}}) = \tilde{G} \cdot \tilde{\mathsf{R}} \subset \tilde{\mathcal{V}},$$

so we deduce that $X \in G \cdot \mathsf{R} \cap \tilde{G} \cdot \tilde{\mathsf{R}} \subset \mathcal{V} \cap \tilde{\mathcal{V}}$.

Now we can choose $g \in G$ such that $g \cdot \mathsf{R} = X$, hence we can conjugate our fixed Cartan involution θ using g, and therefore work with R instead of X. Thus we may assume w.l.o.g that $X := \mathsf{R}$. Now R leaves invariant both V and $\tilde{V}$, in particular implying that:

$$\mathsf{R}(V \cap \tilde{V}) \subset V \cap \tilde{V} \quad \textit{and} \quad \mathsf{R}(V \cap i\tilde{V}) \subset V \cap i\tilde{V}.$$

However again by the compatibility of V and $\tilde{V}$ in $V^{\mathbb{C}}$, we know that $V \cap \tilde{V} = \mathfrak{t}$ and $V \cap i\tilde{V} = \mathfrak{p}$ are the compact/non-compact parts respectively w.r.t our fixed Cartan involution θ. So R and θ commute: $[\mathsf{R}, \theta] = 0$, which immediately implies that $\mathsf{R}_+ = \mathsf{S}$ and $\mathsf{R}_- = \mathsf{A}$ w.r.t θ as required. The theorem is proved.

□

In the case of the Riemann tensor, this is symmetric as a bivector operator with respect to the metric, so we have $\mathsf{R} = \mathsf{S}$, consequently, we get the immediate corollary:

Corollary 3.2. *A metric (of arbitrary dimension and signature) allowing for a standard Wick-rotation at a point p to a Riemannian metric, has a purely electric Riemann tensor, and is consequently purely electric, at p.*

We note that this result applies for a general classes of Wick-rotatable metrics. For example, by complexification of the Lie algebras, it is possible to include Wick-rotations between all of the spaces: de Sitter (dS), anti-de Sitter (AdS), the Riemannian sphere (S^n), and hyperbolic space, (H^n). These are all group quotients G/H of different groups G and H. This seems at first sight paradoxical since these have different signs of the curvature. Thus if $\mathsf{R} = g^{-1} \cdot \tilde{\mathsf{R}}$ as claimed in the proof, they would necessarily have the same Ricci scalar[8] . To understand this we first note that when we Wick-rotate to a Riemannian space we may risk to get either a positive definite metric, $\boldsymbol{g}(v, v) \geq 0$, or a negative definite metric, $\boldsymbol{g}(v, v) \leq 0$. The overall sign is conventional and we say that switching the sign using the "anti-isometry", $\boldsymbol{g} \mapsto -\boldsymbol{g}$ is a matter of convention. Note that this switch of the metric gives the same metric for the metric induced by $\boldsymbol{g}$ on the Lie algebra.

Consider the simple example of the complex holomorphic metric

$$\boldsymbol{g}_{\mathbb{C}} = \frac{1}{(1 + z_1^2 + ... + z_n^2)^2} \left[dz_1^2 + ... + dz_n^2 \right] \tag{3}$$

[6] A vector $X \in V$ is minimal if the norm function $|| - || := \sqrt{\langle -, - \rangle_\theta}$ along an orbit attains a minimum at X; i.e., $||X|| \leq ||h \cdot X||$, $\forall h \in G$.

[7] The conjugation maps of V and $\tilde{V}$ in $V^{\mathbb{C}}$ commute: $\sigma : V^{\mathbb{C}} \to V^{\mathbb{C}}$ and $\tilde{\sigma} : V^{\mathbb{C}} \to V^{\mathbb{C}}$, with $[\sigma, \tilde{\sigma}] = 0$.

[8] The Riemann endomorphism has components related to the Riemann tensor in $T_pM \otimes T_p^*M \otimes (T_pM \otimes T_p^*M)^*$, i.e., $R^{\alpha}{}_{\beta\gamma}{}^{\delta}$. Thus the Ricci scalar is obtained by taking the double trace showing the Ricci scalar is the same after Wick-rotating.

Locally, the two real slices $(z_1, ..., z_n) = (x_1, ..., x_n)$, and $(z_1, ..., z_n) = (iy_1, ..., iy_n)$, give a neighbourhood of S^n and H^n respectively. However, note that for hyperbolic space, the induced metric has the "wrong" sign (it is negative definite). Therefore, considering for example the Ricci tensor (by lowering indices appropriately), we get $R_{\mu\nu} = \lambda g_{\mu\nu}$, $\lambda > 0$, for both real slices, and the sign of the curvature is encaptured in whether the metric is positive or negative definite.

4 Discussion

Using techniques from real invariant theory we have considered a class of metrics allowing for a complex Wick-rotation to a Riemannian space. We have showed that these necessarily are rescricted, in particular, they are purely electric. The result is independent of dimension and signature and shows that if such a Wick rotation is allowable, then we necessarily restrict ourselves to classes of spaces where the "magnetic" degrees of freedom have to vanish (at the point p).

There are many examples of purely electric spaces (see [9, 6] and references therein). In particular, a purely electric Lorentzian spacetime is of type G, I_i, D or O [9]. Thus spacetimes not of these types provide with examples of spaces where such a Wick rotation is not allowed. Non-Wick-rotatable metrics include the classes of Kundt metrics [13] in Lorentzian geometry, and the Walker metrics [14] of more general signature. Also the metrics considered in [15] are in general non-Wick-rotatable metrics. Note that the plane-wave metrics are non-Wick-rotatable metrics.

These results have profound consequences for quantization frameworks where such Wick-rotation is used, since they give a clear restriction of the class of metrics that allows for such a Wick rotation. Clearly, also in the context of quantum gravity, the (real) gravitational degrees of freedom will be restricted by assuming the existence of such a Wick-rotation.

It is worth mentioning that there are quantization procedures which work in the Lorentzian signature all the way through, in particular, there is the algebraic approach to QFT on curved spacetime [16, 17]. For details on renormalization in Lorentzian signature (without Wick rotation), see e.g., [18].

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